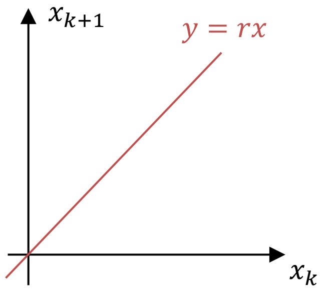
One-dimensional maps

A natural starting point to study nonlinearity is iterations $x_{k+1} = f(x_k)$ with a non-linear function of one variable, a one-dimensional map.

These iterations can also be seen as discretized differential equations:

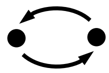


$$\dot{x} = g(x,t) \;\rightarrow\; \frac{\Delta x}{\Delta t} = g(x,t) \;\rightarrow\; x_{t+1} - x_t = g(x_t, t) \;\rightarrow\; x_{t+1} = f(x_t, t)$$

A simple and linear starting point is $x_{k+1} = r x_k$ with solution $x_n = r^n x_0$.

$|r| < 1$ Iteration converges towards $x_\infty = 0$.
$r = 1$ Iteration stuck at $x_n = x_0$, a loop of period 1.

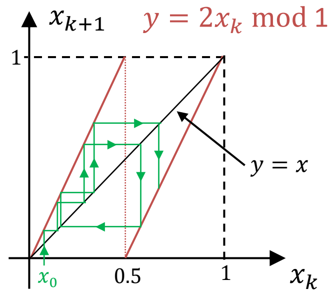
$r = -1$ Iteration alternates $x_n = (-1)^n x_0$, a loop of period 2.



$|r| > 1$ Iteration diverges $|x_n| \rightarrow \infty$.

Iterations with starting points x_0 and y_0 that are initially close $y_0 - x_0 = \delta_0$ will separate at an exponential rate $\delta_n = y_n - x_n = r^n x_0 - r^n y_0 = r^n \delta_0 = e^{n \cdot \ln r} \cdot \delta_0$

The rate of separation for an iteration or a differential equation with $|\delta \boldsymbol{x}(t)| \approx e^{\lambda t} |\delta \boldsymbol{x}(0)|$ is measured by the parameter λ called the **Lyapunov exponent**. In our example $\lambda = \ln r$.
$\lambda < 0$: Convergence at an exponential rate toward a limit or a cycle.
$\lambda > 0$: Divergence at an exponential rate $\delta_n \approx e^{\lambda n} \cdot \delta_0$



Sensitive dependence on initial conditions (SIC) a.k.a. the **butterfly effect** is in the linear setting no more interesting than the exponential function $f(x) = e^x$.

A nonlinear example of the butterfly effect is the **Bernoulli shift**:
$x_{k+1} = 2x_k \bmod 1$

With x_k in binary form, you shift all digits one step to the left and then cut of the integer part.
0.01011011011101 … → 0.1011011011101 … → 0.011011011101 … → …

If $|y_0 - x_0| < 2^{-k}$ it will only take k steps into the iteration and $\boldsymbol{x}$ and $\boldsymbol{y}$ will go separate ways.
This is the butterfly effect, small differences will be amplified and cause great effect.

The iteration $x_{k+1} = f(x_k)$ can be illustrated in a **cobweb diagram** where $y = x_{k+1}$ is taken back to the x-axis via a horizontal displacement towards the line $y = x$.

Of special significance for an iteration $x_{k+1} = f(x_k)$ are points where $f(x^*) = x^*$, so called fixed points. Once a fixed point is reached the iteration gets nowhere.

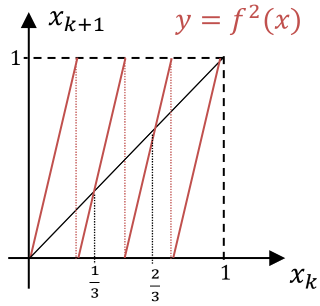
To iterate two steps ahead you apply $f\big(f(x)\big) = f^2(x)$, for n steps ahead you apply $f^n(x) = f\big(f^{n-1}(x)\big)$. If $f^n(x^*) = x^*$ and $x^*, f(x^*), \dots, f^{n-1}(x^*)$ are all unique , i.e. no fixed point $f^k(x^*) = x^*$ if $k|n$ then there is a **period-*k* cycle**. The elements $x^*, \dots, f^{n-1}(x^*)$ form an **orbit**.

The point $0.010101 \ldots = 0.\overline{01}$ is a point of a period-2 cycle for the Bernoulli map $x_{k+1} = 2x_k \bmod 1$.

$$x_0 = 0.\overline{01} \rightarrow (2^2 - 1)x_0 = 1 \rightarrow x_0 = 1/3 \rightarrow x_k = \begin{cases} 1/3 \text{ if } k \text{ is even} \\ 2/3 \text{ if } k \text{ is odd} \end{cases}$$

Any $x_0 \in [0,1]$ that ends with a repeating sequence of digits of length p will end up in an orbit of period p.
$x_0 \in \mathbb{Q} \Rightarrow$ Iteration will end up in a cycle and if x_0 is irrational ($x_0 \in \mathbb{R}\backslash\mathbb{Q}$) it will not end up in a cycle.

Points in orbits of period p are found as intersections of $y = f^p(x)$ and $y = x$.
The graph of $f^p(x)$ for $f(x) = 2x \bmod 1$ has 2^p spikes.



If p is prime there are $2^p - 2$ fixed points of p that are not fixed points of f.
They all belong to an orbit of length p, so $(2^p - 2)/p$ different period-p cycles.

A single fixed point or an orbit can be attraction or repelling for points close to the point or orbit. Start with an orbit of period p: $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_p, \ldots$ with $\bar{x}_p = \bar{x}_0$.

Start from an initial point close to $\bar{x}_0$, $x_0 = \bar{x}_0 + \delta_0$:
$x_p = f^p(\bar{x}_0 + \delta_0)$ Taylor-expand to first order in $\delta_0 \rightarrow \delta_p = \alpha_p \delta_0$ with
$\alpha_p = D(f^p)(\bar{x}_0) = f'(\bar{x}_0) \cdot f'(\bar{x}_1) \cdot \ldots \cdot f'(\bar{x}_{p-1})$ [Repeated use of: $Df^2(x) = f'(f(x)) \cdot f'(x)$]
α is the same for all points in a cycle, permute the points in the product cyclically.

After $2p$ iterations: $\bar{x}_j + \delta_{2p} = \bar{x}_j + \alpha_p \delta_p = \bar{x}_j + \alpha_p^2 \delta_0 \rightarrow$ After mp iterations $\delta_{mp} = \alpha_p^m \delta_0$

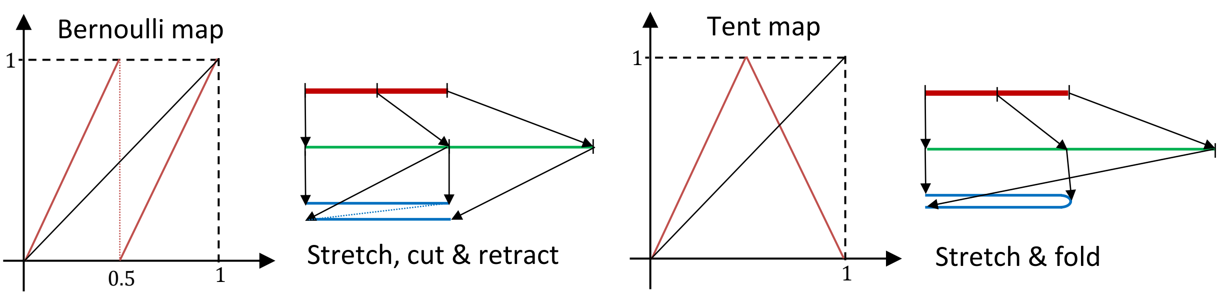
α is called the **stability coefficient**.
Deviation from the orbit grows if $|\alpha| > 1$, these orbits are called **repelling** or **unstable**.
Deviation from the orbit shrinks if $|\alpha| < 1$, these orbits are called **attracting** or **stable**.
Orbits with $\alpha = 0$ are called **super stable**.

The Bernoulli map has $f'(x) = 1/2$ if $x \neq 1/2 \rightarrow \alpha = 2^p$ for a period-p orbit, very repelling.

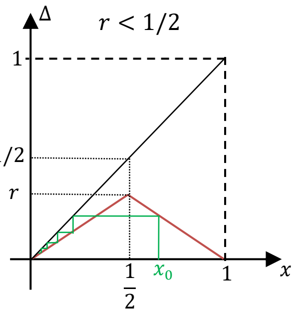
The Lyapunov exponent that measures exponential separation $\delta \rightarrow \delta e^{n\lambda}$ (after n iterations) is closely related to the stability coefficient, $\delta e^{n\lambda(x_0)} \approx |f^n(x_0 + \delta) - f^n(x_0)|$.

$$\lambda(x_0) = \lim_{n\to\infty} \lim_{\delta\to 0} \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta) - f^n(x_0)}{\delta} \right| = \lim_{n\to\infty} \frac{1}{n} |Df^n(x_0)| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln|f'(x_k)|$$

A variation of the Bernoulli map but without discontinuity is the **tent map** $x_{k+1} = 1 - |1 - 2x_k|$.



The tent map iteration resembles the multi-layered treatment of dough in a danish pastry.
With a parameter r for the height of the tent it is a good illustration of order and chaos.

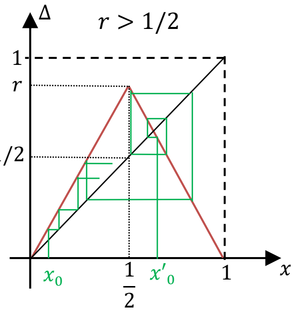


Tent map

$\Delta(x) = r(1 - |1 - 2x|)$

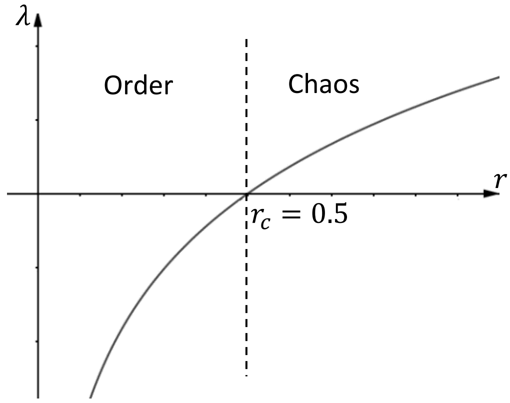
$|\Delta'(x)| = 2r$

$\lambda(x) = \ln 2r$

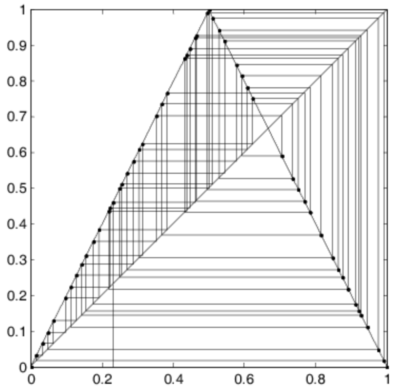


For $r < 0.5$ there is one attracting fixpoint, $x^* = 0$ that attracts all $x_0 \in [0,1]$.
For $r > 0.5$ there are two repelling fixpoints, $x^* = 0$ and $x^* = 2r/(2r+1)$.

The attracting fixpoint splits into two repelling fixpoints when the Lyapunov exponent changes sign.
r acts as an **order parameter** with a 'phase change' from order to chaos at the **critical point** $r_c = 0.5$.



Chaos in tent map:
$x_{k+1} = \Delta(x_k)$
$= 1 - |1 - 2x_k|$
when $r > r_c = 0.5$



Chaos in a mathematical setting means a system with random behaviour that is ruled by deterministic laws. Properties associated with chaos are sensitive dependence on initial conditions (positive Lyapunov exponent) and dense, repelling periodic orbits. Dense means that every point is arbitrarily close to a periodic orbit.

$\mathbb{Q}$ is dense in $\mathbb{R}$ and every $x_0 \in \mathbb{Q}$ of the Bernoulli map ends up in a periodic orbit but
no starting point in $\mathbb{R}\backslash\mathbb{Q}$ of the Bernoulli map will enter or approach a periodic orbit.

Another property common in chaotic systems is **ergodicity**. The stretch and fold property of the tent map means that a typical spice corn in a dough will eventually come arbitrarily close to any position in the dough, in a way that favours no particular position. This is the ergodic property.

Ergodicity can occur in a physical system when geodesics diverges on a compact manifold. Solutions to the dynamical equations that govern a system can produce orbits that eventually fills the available space,
like smoke that enters a room or temperature that eventually become uniform as equilibrium is approached.

The ergodic hypothesis of statistical physics states that over long periods of time,
the time spent in a region of phase space is proportional to the volume of the region.
All microstates are equally probable to be occupied in the long run.